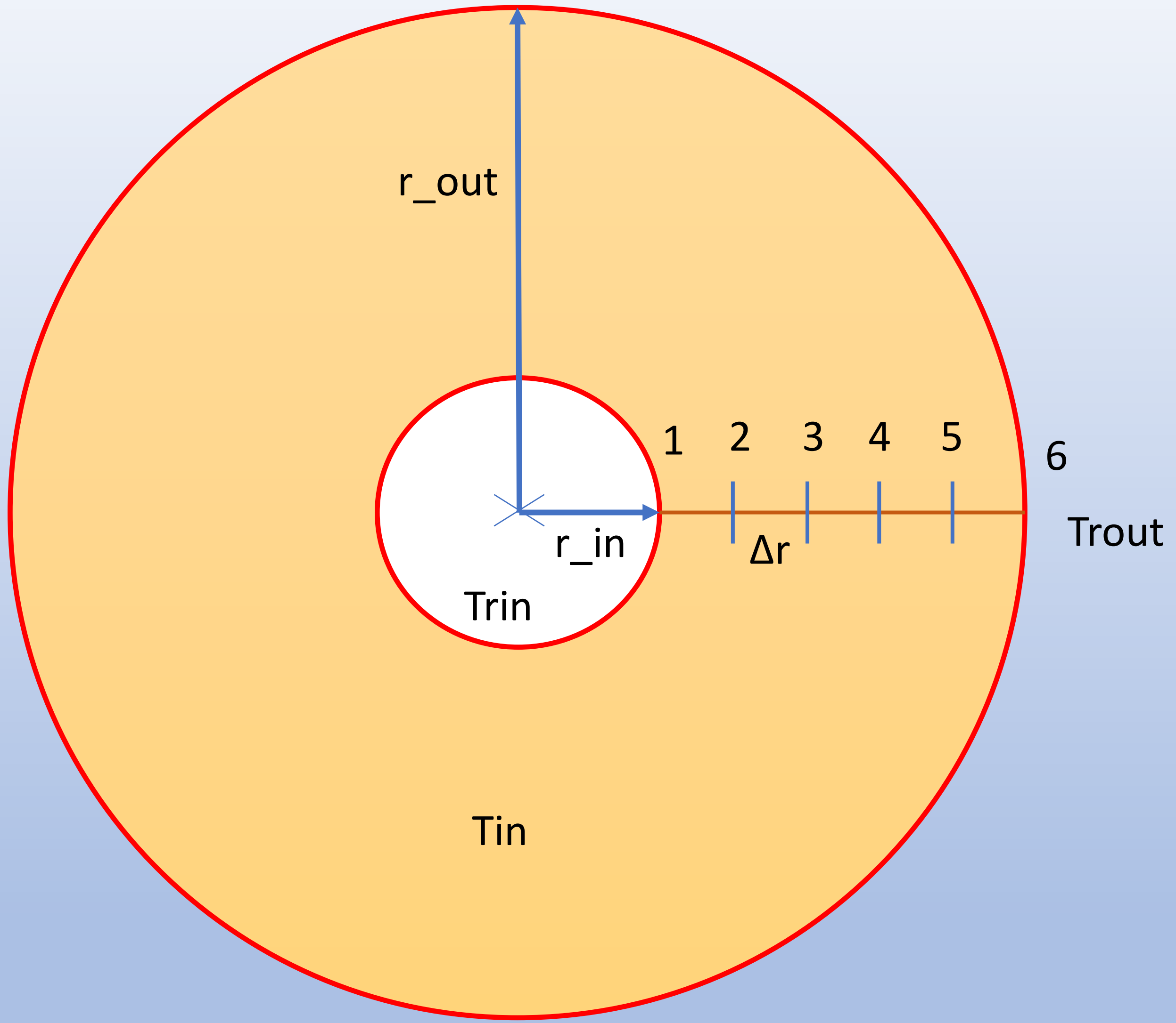
Solve 1D Transient Heat Conduction Problem in Cylindrical Coordinates Using Finite Difference Method

Objectives

- Present a simple 1D Transient heat conduction problem in cylindrical coordinates
- Temperature variation is along the radial direction
- Solve the problem using (FTCS) FDM
- Vary grid spacings and obtain solutions using FDM and present the results graphically



DOMAIN

General Heat Conduction Equation in 3D Cylindrical Coordinates

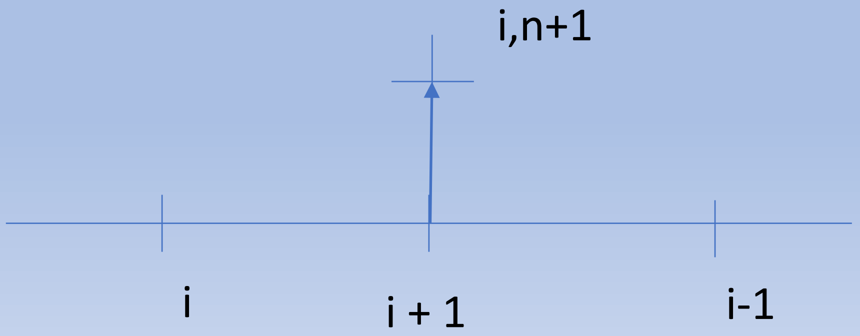
- $\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right)\left(\frac{\partial T}{\partial r}\right) + \left(\frac{1}{r^2}\right)\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (1) ;

- T = T(r,ϕ,z,t); r – radial dimension, ϕ – phase angle, z – vertical dimension, t – time, s
- $\alpha - Thermal\ diffusivity, \frac{m^2}{s}$
- $\alpha = k/(\rho * c)$
- k – thermal conductivity of the material , W/(m K)
- ρ - density of the material, kg/m^3
- c – specific heat capacity of the material, J/(kg K)
- g – volumetric rate of internal heat generation, $W/(m^3)$
- We assume that k is uniform in the domain (homogeneous and isotropic)

For 1D transient heat conduction along radial direction with no heat generation, equation (1) reduces to a simpler form

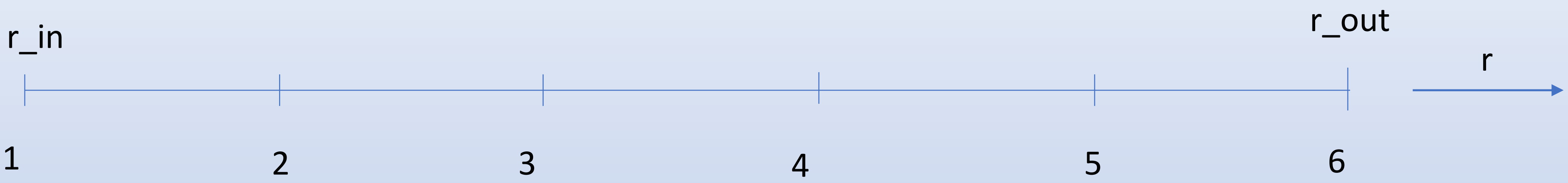
- We assume temperature does not vary significantly in ϕ, z direction when compared with radial direction.
- g = 0 (no heat generation);
- $\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right)\left(\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (2); Rearranging the equation, we get,
- $\frac{\partial T}{\partial t} = \alpha * \left(\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right)\left(\frac{\partial T}{\partial r}\right)\right)$; T = T(r, t); (3)
- IC: T(r, t = 0) = Tin
- BCs: At r = r_in, T = Trin ; At r = r_out, T = Trout;
- To obtain T, we solve the above PDE using Finite Difference Method .
- To do so, we need to replace the derivatives with finite difference approximations
- We will replace both the first and second order space derivatives with centered difference approximations and the time derivative with forward difference approximation.

- $\left(\frac{T_i^{n+1} - T_i^n}{\Delta t}\right) = \alpha * \left(\left(\frac{T_{i-1}^n - 2*T_i^n + T_{i+1}^n}{\Delta r^2}\right) + \left(\frac{1}{r_i}\right) * \left(\frac{T_{i+1}^n - T_{i-1}^n}{2 * \Delta r}\right)\right)$
- Rearranging the above equation we get,
- $T_i^{n+1} - T_i^n = \left(\left(\frac{\alpha * \Delta t}{\Delta r^2}\right)(T_{i-1}^n - 2 * T_i^n + T_{i+1}^n) + \left(\frac{\alpha * \Delta t}{2 * \Delta r * r_i}\right) * (T_{i+1}^n - T_{i-1}^n)\right)$
- Here d = $\left(\frac{\alpha * \Delta t}{\Delta r^2}\right)$; d – diffusion no. ; Let d1 = $\left(\frac{\alpha * \Delta t}{2 * \Delta r}\right)$
- $T_i^{n+1} = T_i^n + d * (T_{i-1}^n - 2 * T_i^n + T_{i+1}^n) + \left(\frac{d1}{r_i}\right) * (T_{i+1}^n - T_{i-1}^n)$ (4)
- Equation (4) is the finite difference approximation of the original PDE equation we were trying to solve.
- Here i represents the node location and n the time step on the discretized domain.
- The finite difference stencil is given below.



- The above approximation is called Forward-Time Centered-Space (FTCS) method.
- This is an explicit method. Hence, temperatures T_i's at future times (n+1) can be directly obtained based on T_i's at present times as shown in equation (4)
- Explicit methods are conditionally stable.
- The stability criteria is given as $d <= 0.5$
- Also, diffusion number (d) / time step (Δt) needs to be smaller based on the accuracy desired
- The error is of $O(\Delta t) + O(\Delta r^2)$

- Now let us discretize the 1D domain into say 5 segments / grid spacings (equally spaced) as shown below.
- Note Temperatures at Node 1) and Node 6) are known (BCs).



- Note interior nodes are 2 to 5 and boundary nodes are 1 and 6.
- Let r_in = 0.1 m; r_out = 1.1 m; Trin = 200 °C; Trout = 300 °C; Tin = 100 °C
- No. of segments, m = 5; total time t = 200s; time step Δt = 100 s; so, no. of time steps nt = 2;
- $\Delta r = \left(\frac{r\_out - r\_in}{m}\right) = \left(\frac{1.1 - 0.1}{5}\right) = 0.2$ m
- Hence $r_2 = 0.3$ m, $r_3 = 0.5$ m, $r_4 = 0.7$ m, $r_5 = 0.9$ m
- $d = \left(\frac{\alpha * \Delta t}{\Delta r^2}\right) = \left(\frac{1e{-4} * 100}{0.2^2}\right) = 0.25 < 0.5$ (stability criteria met)
- $d1 = \left(\frac{\alpha * \Delta t}{2 * \Delta r}\right) = \left(\frac{1e{-4} * 100}{2 * 0.2}\right) = 0.025$

- Let i = 2, 3, 4, 5 & n = 0, Eq 4) becomes
- $T_2^1 = T_2^0 + \mathrm{d} * (T_1^0 - 2 * T_2^0 + T_3^0) + \left(\frac{d1}{r_2}\right) * (T_3^0 - T_1^0)$
- $T_3^1 = T_3^0 + \mathrm{d} * (T_2^0 - 2 * T_3^0 + T_4^0) + \left(\frac{d1}{r_3}\right) * (T_4^0 - T_2^0)$
- $T_4^1 = T_4^0 + \mathrm{d} * (T_3^0 - 2 * T_4^0 + T_5^0) + \left(\frac{d1}{r_4}\right) * (T_5^0 - T_3^0)$
- $T_5^1 = T_5^0 + \mathrm{d} * (T_4^0 - 2 * T_5^0 + T_6^0) + \left(\frac{d1}{r_5}\right) * (T_6^0 - T_4^0)$

- Substituting the values T_i^0s for the interior nodes and T_1 and T_6 values for the boundary nodes, d, d1 and r_is into the above equation, we get,

- $T_2^1 = 100 + 0.25 * (200 - 2 * 100 + 100) + \left(\frac{0.025}{0.3}\right) * (100 - 200)$
- $T_3^1 = 100 + 0.25 * (100 - 2 * 100 + 100) + \left(\frac{0.025}{0.5}\right) * (100 - 100)$
- $T_4^1 = 100 + 0.25 * (100 - 2 * 100 + 100) + \left(\frac{0.025}{0.7}\right) * (100 - 100)$
- $T_5^1 = 100 + 0.25 * (100 - 2 * 100 + 300) + \left(\frac{0.025}{0.9}\right) * (300 - 100)$
- Solving the above equation we get the following results.
- T2 = 116.67 °C, T3 = 100 °C, T4 = 100 °C, T5 = 155.56 °C

- For the second time step, i = 2, 3, 4, 5 & n = 1, Eq 4) becomes

- $T_2^2 = T_2^1 + \mathrm{d} * (T_1^1 - 2 * T_2^1 + T_3^1) + \left(\frac{d1}{r_2}\right) * (T_3^1 - T_1^1)$

- $T_3^2 = T_3^1 + \mathrm{d} * (T_2^1 - 2 * T_3^1 + T_4^1) + \left(\frac{d1}{r_3}\right) * (T_4^1 - T_2^1)$

- $T_4^2 = T_4^1 + \mathrm{d} * \left(T_3^1 - 2 * T_4^1 + T_5^1\right) + \left(\frac{d1}{r_4}\right) * \left(T_5^1 - T_3^1\right)$

- $T_5^2 = T_5^1 + \mathrm{d} * \left(T_4^1 - 2 * T_5^1 + T_6^1\right) + \left(\frac{d1}{r_5}\right) * (T_6^0 - T_4^1)$

- Substituting the values T_i^1s for the interior nodes and T_1 and T_6 values for the boundary nodes, d, d1 and r_is into the above equation, we get,

- $T_2^1 = 116.67 + 0.25 * (200 - 2 * 116.67 + 100) + \left(\frac{0.025}{0.3}\right) * (100 - 200)$
- $T_3^1 = 100 + 0.25 * (116.67 - 2 * 100 + 100) + \left(\frac{0.025}{0.5}\right) * (100 - 116.67)$
- $T_4^1 = 100 + 0.25 * (100 - 2 * 100 + 155.56) + \left(\frac{0.025}{0.7}\right) * (155.56 - 100)$
- $T_5^1 = 155.56 + 0.25 * (100 - 2 * 155.56 + 300) + \left(\frac{0.025}{0.9}\right) * (300 - 100)$
- Solving the above equation we get the following results.
- T2 = 125.00 °C, T3 = 103.33 °C, T4 = 115.87 °C, T5 = 183.33 °C

- Graphical results are presented using MATLAB for this case.
- Using MATLAB or other software, we can develop codes for a general case where the number of grid spacings is, say m and the number of time steps is say, nt and solution obtained accordingly.
- We can then vary the number of grid spacings to say 10 and time steps to say, 300 and solve the problem again and present the results graphically.

Summary

In this video,

- We presented a 1D Transient heat conduction problem in Cylindrical Coordinates
- Temperature variation is along the radial direction
- The initial temperature is given and temperatures at the inner and outer surfaces are fixed.
- We solved the problem using (FTCS) Finite difference method and obtained the temperature profile
- We resolved the problem using smaller grid spacings and time steps and presented the results
- In future videos, we can explore more challenging problems.